On Internally Cancellable Rings

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Noncommutative Rings and their Applications
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Outline

Let R be an associative unital ring.
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An element a in a ring R is called *unit-regular* if there exists a unit element $u \in R$ such that $a = aua$. The ring R is called *unit-regular* if every element in R is unit-regular.

Definition [Bass, 1964]

A ring R has *stable range one* ($\mathrm{sr}(R) = 1$) if, for each $a, b \in R$ with $Ra + Rb = R$ there exists $x \in R$ such that $a + xb \in \mathrm{U}(R)$.

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A right R-module M is said to be *internally cancellable* (IC, for short) if $M = M_1 \oplus M_2 = N_1 \oplus N_2$ (in the category of R-modules) and $M_1 \cong N_1$ together imply that $M_2 \cong N_2$.

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The IC-property of rings is right-left symmetric.

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(4) The left analogues of (2) and (3).

Consider the following statement:

$$(\Diamond):\ Ra + Rb = R \text{ implies that } a + xb \in \mathrm{U}(R) \text{ for some } x \in R,$$

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(6) For each $a \in R$ and $b \in \mathrm{Reg}(R)$, $(\diamondsuit)$ holds.

$$R \text{ is unit-regular} \implies sr(R) = 1 \implies rsr(R) = 1 \iff R \text{ is IC}$$

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It is observed that $(*)$ and $(\diamondsuit)$ have different behavior.

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- Next we consider the element a to be regular in $(*)$ whenever $b \in R$, $b \in \mathrm{Reg}(R)$ or $b \in \mathrm{Idem}(R)$ and characterize IC rings.

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For any idempotents e, f in a ring R, if $Re + Rf = R$, then there exists a unit regular element $x \in Rf$ such that $e + xf \in \mathrm{U}(R)$ and $eR \cap xR = 0$.

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Definition [Khurana-Lam, 2005]

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Definition [Garg, Grover and Khurana, 2014]

A ring R is called *perspective* if any two isomorphic direct summands of R have a common complement, i.e. if $eR \cong fR$ for any $e, f \in \mathrm{Idem}(R)$, then there exists a direct summand C of R such that
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- Regular rings and abelian rings have the summand sum property.

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- Hence R is not perspective by [Garg, Grover and Khurana, 2014]
- On the other hand, $\mathbb{Z} \oplus \mathbb{Z}$ has not SSP as a $\mathbb{Z}$-module, hence R has not SSP by [Goodearl, 1991]

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(2) Every regular element of R has idempotent stable range one.

Definition [Abrams-Rangaswamy, 2010]

An element a in R is called *special clean* if there exists a decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \mathrm{Idem}(R)$, $u \in \mathrm{U}(R)$. The ring R is called *special clean* if every element of R is special clean.

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Proposition 3.1

The following are equivalent for a ring R.

(1) R is IC;

(2) For every $a \in \mathrm{Reg}(R)$, there exists $u \in \mathrm{U}(R)$ such that au is special clean.

Theorem [Camillo-Khurana, 2001]

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- This gives the following fact for a ring R:

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Lemma 3.2

Any left non-zero divisor regular element over an abelian ring is a unit.

Theorem 3.3

Let R be an abelian ring. Then for every $a \in \mathrm{Reg}(R)$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \mathrm{Idem}(R)$, $u \in \mathrm{U}(R)$.

Theorem 3.3

Let R be an abelian ring. Then for every $a \in \mathrm{Reg}(R)$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \mathrm{Idem}(R)$, $u \in \mathrm{U}(R)$.

Corollary 3.4 [Akalan-Vaš, 2013]

If R is abelian, then R is unit regular if and only if for every $a \in R$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \mathrm{Idem}(R)$, $u \in \mathrm{U}(R)$.

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